Robust Control of an Observation Satellite Attitude with Parametric Uncertainties

(Odporne Sterowanie Orientacją Satelity Obserwacyjnego przy Niepewnych Parametrach)

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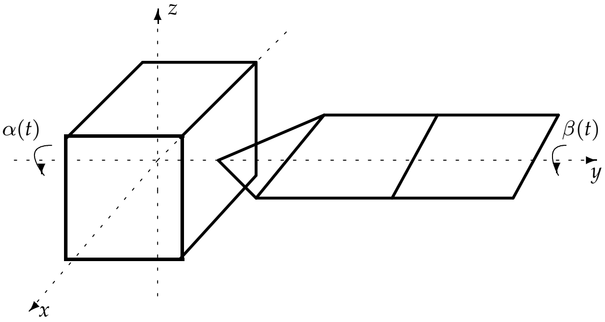

Figure 1: Satellite with a solar panel

Plant equations of motion

- *Plant*

$$\Sigma_G : \begin{cases} I\ddot{\alpha}(t) &= k(\beta(t)-\alpha(t)) + b(\dot{\beta}(t)-\dot{\alpha}(t)) + u(t)\,, \\ p\ddot{\beta}(t) &= -k(\beta(t)-\alpha(t)) - b(\dot{\beta}(t)-\dot{\alpha}(t))\,, \end{cases} \tag{1}$$

with some initial conditions.

- $u(t)$ is an input torque, regarded as an *input*,
- $\alpha(t)$ is a satellite angular displacement, a measured signal regarded as an *output*,
- $\beta(t)$ is a panel angular displacement, an unmeasured signal,
- I is a satellite rotational inertia,
- p is a panel rotational inertia,
- k is a stiffness coefficient,
- b is a friction coefficient.

- Assume: $I > 0$, $p > 0$, $k > 0$ and $b > 0$ (incl. $b = 0$).
- In practice, I, p, k and b cannot be measured exactly, they are *uncertain*.

Uncertain parameters

- *Known intervals*

$$I \in (I_{\min}, I_{\max}), \quad p \in (p_{\min}, p_{\max}), \quad k \in (k_{\min}, k_{\max}), \quad b \in (b_{\min}, b_{\max}).$$

- *Nominal* (mean) values

$$I(0) = \frac{I_{\min} + I_{\max}}{2}, \quad p(0) = \frac{p_{\min} + p_{\max}}{2}, \quad k(0) = \frac{k_{\min} + k_{\max}}{2}, \quad b(0) = \frac{b_{\min} + b_{\max}}{2},$$

- *Weight coefficients*

$$W_I = \frac{I_{\max} - I_{\min}}{2}, \quad W_p = \frac{p_{\max} - p_{\min}}{2}, \quad W_k = \frac{k_{\max} - k_{\min}}{2}, \quad W_b = \frac{b_{\max} - b_{\min}}{2},$$

- Uncertain real parameters in *additive forms*

$$I(\delta_I) = I(0) + W_I\delta_I, \quad p(\delta_p) = p(0) + W_p\delta_p, \quad k(\delta_k) = k(0) + W_k\delta_k, \quad b(\delta_b) = b(0) + W_b\delta_b,$$

where δ_I, δ_p, δ_k and δ_b are *normalized uncertainties*, i.e.

$$|\delta_I| < 1, \quad |\delta_p| < 1, \quad |\delta_k| < 1, \quad |\delta_b| < 1.$$

More explicit form of the plant

- We interpret the nominal parameters $I(0)$, $p(0)$, $k(0)$ and $b(0)$ as *real measurements* and W_I, W_p, W_k and W_b describe *bounds on the errors*.

- It follows

$$\begin{cases} I_{\min} = I(0) - W_I \\ I_{\max} = I(0) + W_I \end{cases}, \quad \begin{cases} p_{\min} = p(0) - W_p \\ p_{\max} = p(0) + W_p \end{cases}, \quad \begin{cases} k_{\min} = k(0) - W_k \\ k_{\max} = k(0) + W_k \end{cases}, \quad \begin{cases} b_{\min} = b(0) - W_b \\ b_{\max} = b(0) + W_b \end{cases}. \tag{2}$$

- The *joint uncertainty*

$$\delta := (\delta_I, \delta_p, \delta_k, \delta_b)\,, \tag{3}$$

- The plant in the *explicit uncertain form*

$$\Sigma_G(\delta): \begin{cases} I(\delta_I)\ddot{\alpha}(t) = k(\delta_k)(\beta(t) - \alpha(t)) + b(\delta_b)(\dot{\beta}(t) - \dot{\alpha}(t)) + u(t)\,, \\ p(\delta_p)\ddot{\beta}(t) = -k(\delta_k)(\beta(t) - \alpha(t)) - b(\delta_b)(\dot{\beta}(t) - \dot{\alpha}(t))\,. \end{cases}$$

- $u(t)$ consists of a *control torque* $\tau(t)$ and a *disturbance torque* $d(t)$, i.e.

$$u(t) = \tau(t) + d(t)\,, \quad t \geq 0\,, \tag{4}$$

where

$$d(t) = d_0 = \text{const}\,, \quad t \geq 0\,, \tag{5}$$

with an *unknown* magnitude $d_0 \in \mathbb{R}$.

Formulation of the control problem

- *Reference signal*

$$\alpha_r(t) = a \sin \omega_r t\,, \quad a > 0\,, \quad \omega > 0\,, \tag{6}$$

- *Control error*

$$e(t) = \alpha(t) - \alpha_r(t)\,. \tag{7}$$

- *Control goal* (asymptotic tracking)

$$\lim_{t\to\infty} e(t) = 0 \tag{8}$$

  for all disturbances $d_0 \in \mathbb{R}$.

- *Dynamic error feedback controller*

$$\begin{bmatrix} \dot{x}_K(t) \\ \tau(t) \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K(t) \\ e(t) \end{bmatrix}, \tag{9}$$

  where $(x_K(t))_{t\geq 0} \subset \mathbb{R}^{n_K}$ and $e(t)$ is the only signal available to the controller.

- *Unit feedback control system* (Figure 2)

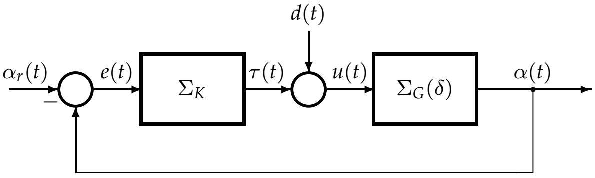

Figure 2: Error feedback control system

Uncertain plant state space model

- *State variables*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \tag{10}$$

- *Plant state space model*

$$\Sigma_G(\delta): \quad \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \hline \alpha \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\dfrac{k(\delta_k)}{I(\delta_I)} & \dfrac{k(\delta_k)}{I(\delta_I)} & -\dfrac{b(\delta_b)}{I(\delta_I)} & \dfrac{b(\delta_b)}{I(\delta_I)} & \dfrac{1}{I(\delta_I)} \\ \dfrac{k(\delta_k)}{p(\delta_p)} & -\dfrac{k(\delta_k)}{p(\delta_p)} & \dfrac{b(\delta_b)}{p(\delta_p)} & -\dfrac{b(\delta_b)}{p(\delta_p)} & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hline u \end{array}\right], \tag{11}$$

- Simplified state space model of the uncertain plant

$$\Sigma_G(\delta): \quad \left[\begin{array}{c} \dot{x} \\ \hline \alpha \end{array}\right] = \left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline u \end{array}\right], \tag{12}$$

- $\Sigma_G(\delta)$ is referred to as the *uncertain plant model.*

Nominal plant state space model

- Plant state space model with $\delta = 0$

$$\Sigma_G(0): \quad \left[\begin{array}{c} \dot{x} \\ \hline \alpha \end{array}\right] = \left[\begin{array}{c|c} A(0) & B(0) \\ \hline C & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline \tau \end{array}\right], \tag{13}$$

where

$$\left[\begin{array}{c|c} A(0) & B(0) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k(0)}{I(0)} & \frac{k(0)}{I(0)} & -\frac{b(0)}{I(0)} & \frac{b(0)}{I(0)} & \frac{1}{I(0)} \\ \frac{k(0)}{p(0)} & -\frac{k(0)}{p(0)} & \frac{b(0)}{p(0)} & -\frac{b(0)}{p(0)} & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \tag{14}$$

- $\Sigma_G(0)$ is referred to as the *nominal plant model*.
- $\Sigma_G(\delta)$ is *controllable* and *observable*

$$\det W(\delta) = -\frac{(k(\delta_k))^2}{(I(\delta_I))^4 (p(\delta_p))^2} \neq 0, \quad \det V(\delta) = -\frac{(k(\delta_k))^2}{((\delta_I))^2} \neq 0, \tag{15}$$

for all δ_I, δ_p, δ_k and δ_b.

- In particular, the nominal plant $\Sigma_G(0)$ is also controllable and observable.

Two dynamical systems

- *Reference signal* $\alpha_r(t) = a\sin(\omega_r t + \varphi)$ is *generated by a dynamical system*

$$\begin{bmatrix} \dot{r}_1(t) \\ \dot{r}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_r^2 & 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \quad \begin{bmatrix} r_1(0) \\ r_2(0) \end{bmatrix} = \begin{bmatrix} a\sin\varphi \\ a\omega\cos\varphi \end{bmatrix}, \tag{16}$$

  and

$$\alpha_r(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}, \tag{17}$$

  where $\omega_r > 0$ has to be known and $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$ may be unknown.

- The *disturbance* $d(t) = d_0$ is *generated by a dynamical system*

$$\dot{d}(t) = 0 \cdot d(t)\,, \quad d(0) = d_0\,, \tag{18}$$

  and

$$d(t) = 1 \cdot d(t)\,, \tag{19}$$

  where $d_0 \in \mathbb{R}$ is unknown.

Exosystem

- By combining (16)-(19) we get a dynamical system Σ_S, called the *exosystem*,

$$\Sigma_S \;:\; \left[\begin{array}{c} \dot{r}_1 \\ \dot{r}_2 \\ \dot{d} \\ \hline \alpha_r \\ d \end{array}\right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ -\omega_r^2 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} r_1 \\ r_2 \\ d \end{bmatrix}, \quad \begin{bmatrix} r_1(0) \\ r_2(0) \\ d(0) \end{bmatrix} = \begin{bmatrix} a\sin\varphi \\ a\omega_r\cos\varphi \\ d_0 \end{bmatrix}, \tag{20}$$

i.e.

$$\Sigma_S \;:\; \left[\begin{array}{c} \dot{w} \\ \hline \alpha_r \\ d \end{array}\right] = \left[\begin{array}{c} S \\ \hline T_r \\ T_d \end{array}\right] w\,, \quad w(0) = w_0\,, \tag{21}$$

where

$$w := \begin{bmatrix} r_1 \\ r_2 \\ d \end{bmatrix}, \tag{22}$$

with eigenvalues (the spectrum)

$$\sigma(S) = \{0; j\omega_r; -j\omega_r\}\,. \tag{23}$$

- $\sigma(S) \cap \mathbb{C}_- = \emptyset$.

Robust control system

- *Uncertainty matrix*

$$\Delta(\delta) = \begin{bmatrix} \delta_k & 0 & 0 & 0 \\ 0 & \delta_b & 0 & 0 \\ 0 & 0 & \delta_I & 0 \\ 0 & 0 & 0 & \delta_p \end{bmatrix} = \begin{bmatrix} \delta_k + j0 & 0 & 0 & 0 \\ 0 & \delta_b + j0 & 0 & 0 \\ 0 & 0 & \delta_I + j0 & 0 \\ 0 & 0 & 0 & \delta_p + j0 \end{bmatrix}, \qquad (24)$$

- *Uncertainty structure set* $\Delta_c \subset \mathbb{C}^{4\times 4}$

$$\Delta_c := \{\Delta(\delta) \in \mathbb{C}^{4\times 4} : \ \sigma_{\max}(\Delta(\delta)) < 1 \ \} \qquad (25)$$

- Uncertain plant $\Sigma_G(\delta)$

$$\Sigma_G(\delta) : \left\{ \begin{array}{rcl} \dot{x} & = & A(\delta)x + B(\delta)u \,, \quad x(0) = x_0 \,, \\ \alpha & = & Cx \,, \end{array} \right. \qquad \Delta(\delta) \in \Delta_c \,, \qquad (26)$$

where $u = \tau + d$.

- *Controller* and *exosystem*

$$\Sigma_K : \left\{ \begin{array}{rcl} \dot{x}_K & = & A_K x_K + B_K e \,, \quad x_K(0) = x_{K0} \,, \\ \tau & = & C_K x_K + D_K e \,, \end{array} \right. \quad \Sigma_S : \left\{ \begin{array}{rcl} \dot{w} & = & Sw \,, \quad w(0) = w_0 \,, \\ \alpha_r & = & T_r w \,, \\ d & = & T_d w \,. \end{array} \right. \qquad (27)$$

- *Error*

$$e = \alpha - \alpha_r \,.$$

Error feedback control system

- $\Sigma_G(\delta)$ and Σ_K gives the *error feedback control system* $\Sigma_e(\delta)$

$$\Sigma_e(\delta):\left[\begin{array}{c}\dot{x}\\ \dot{x}_K\\ \hline e\end{array}\right]=\left[\begin{array}{cc|cc}A(\delta)+B(\delta)D_KC & B(\delta)C_K & -B(\delta)D_K & B(\delta)\\ B_KC & A_K & -B_K & 0\\ \hline C & 0 & -I & 0\end{array}\right]\left[\begin{array}{c}x\\ x_K\\ \hline \alpha_r\\ d\end{array}\right],\quad \Delta(\delta)\in\Delta_c\,. \tag{28}$$

- The *unforced closed loop system* $\Sigma_{uf}(\delta)$ ($\alpha_r\equiv 0, d\equiv 0$)

$$\Sigma_{uf}(\delta):\left[\begin{array}{c}\dot{x}\\ \dot{x}_K\end{array}\right]=\left[\begin{array}{cc}A(\delta)+B(\delta)D_KC & B(\delta)C_K\\ B_KC & A_K\end{array}\right]\left[\begin{array}{c}x\\ x_k\end{array}\right],\quad \Delta(\delta)\in\Delta_c\,. \tag{29}$$

- Interconnection of $\Sigma_e(\delta)$ and Σ_S gives:
- The *closed loop system* $\Sigma_{cl}(\delta)$

$$\Sigma_{cl}(\delta):\left[\begin{array}{c}\dot{x}\\ \dot{x}_K\\ \dot{w}\\ \hline e\end{array}\right]=\left[\begin{array}{ccc}A(\delta)+B(\delta)D_KC & B(\delta)C_K & B(\delta)(T_d-D_KT_r)\\ B_KC & A_K & -B_KT_r\\ 0 & 0 & S\\ \hline C & 0 & -T_r\end{array}\right]\left[\begin{array}{c}x\\ x_K\\ w\end{array}\right],\quad \Delta(\delta)\in\Delta_c\,. \tag{30}$$

Precise requirements

The error feedback controller Σ_K is to guarantee:

RIS: *Robust internal stability*. The error feedback control system $\Sigma_e(\delta)$ is said to be *robustly internally stable* if the unforced closed system $\Sigma_{uf}(\delta)$ is *asymptotically stable* for all $\Delta(\delta) \in \Delta_c$, i.e. for all $x(0) = x_0$, $x_K(0) = x_{K0}$ we have

$$\lim_{t \to \infty} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} = 0\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{31}$$

RAT: *Robust asymptotic tracking* (called *robust regulation*). The error feedback control system $\Sigma_e(\delta)$ is said to be is said to satisfy the *robust asymptotic tracking* condition if for all $w(0) = w_0$, $x(0) = x_0$ and $x_K(0) = x_{K0}$ the closed loop system $\Sigma_{cl}(\delta)$ satisfies

$$\lim_{t \to \infty} e(t) = 0\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{32}$$

- Every controller Σ_K which guarantees RIS and RAT is said to be a *robust controller*.
- Is is seen from (29) that RIS holds if and only if

$$\sigma\left(\begin{bmatrix} A(\delta) + B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{bmatrix} \right) \subset \mathbb{C}_-\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{33}$$

- Examination of RIS is a hard task and will be dealt with later on.
- Before that, we show how to deal with RAT under the assumption that RIS holds.

Fundamental result

- Since $\Sigma_G(\delta)$ is controllable and observable for all $\Delta(\delta) \in \Delta_c$, then there always exists a controller (A_K, B_K, C_K, D_K) (possibly dependent of δ) satisfying

$$\sigma\left(\begin{bmatrix} A(\delta) + B(\delta)D_K C & B(\delta)C_K \\ B_K C & A_K \end{bmatrix}\right) \subset \mathbb{C}_- ,$$

- For a controller (A_K, B_K, C_K, D_K), *independent* of δ and *satisfying* RIS, we get necessary and sufficient conditions for RAT:

Theorem 3.1

If for a given controller (A_K, B_K, C_K, D_K) RIS holds, then $\Sigma_e(\delta)$ satisfies RAT if and only if there exist $\Pi(\delta) \in \mathbb{R}^{4\times 3}$, $\Gamma(\delta) \in \mathbb{R}^{1\times 3}$ and $\Sigma(\delta) \in \mathbb{R}^{n_K\times 3}$ such that

$$RE : \begin{cases} A(\delta)\Pi(\delta) - \Pi(\delta)S + B(\delta)\Gamma(\delta) + B(\delta)T_d = 0 \,, \\ C\Pi(\delta) - T_r = 0 \,, \end{cases} \tag{34}$$

and

$$IMP : \begin{cases} \Gamma(\delta) = C_K\Sigma(\delta) \,, \\ \Sigma(\delta)S = A_K\Sigma(\delta) \,, \end{cases} \tag{35}$$

for all $\Delta(\delta) \in \Delta_c$. If this is the case, then (A_K, B_K, C_K, D_K) is a robust controller.

- RE stands for the *regulator equation* and IMP for the *internal model principle*.

Construction of a robust controller

- RE has a solution $(\Pi(\delta), \Gamma(\delta))$.
- Define

$$P := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_r^2 & 0 \end{bmatrix}, \quad R := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- For every controller (A_K, B_K, C_K, D_K) of order n_K, which is independent of δ and has the form

$$\begin{aligned} A_K &= \begin{bmatrix} P & 0 \\ 0 & A_v \end{bmatrix} \in \mathbb{R}^{n_K \times n_K}, \quad & B_K &= \begin{bmatrix} Q \\ B_v \end{bmatrix} \in \mathbb{R}^{n_K \times 1}, \\ C_K &= \begin{bmatrix} R & C_v \end{bmatrix} \in \mathbb{R}^{1 \times n_K}, & D_K &= D_v \in \mathbb{R}^{1 \times 1}, \end{aligned} \tag{36}$$

  where A_v, B_v, C_v, D_v are arbitrary, there always exists $\Sigma(\delta) \in \mathbb{R}^{n_K \times 3}$ such that IMP *holds*.
- The controller Σ_K consists of two paralel systems

$$\Sigma_w : \begin{cases} \dot{w} &= Pw + Qe\,, \\ y_w &= Rw\,, \end{cases} \qquad \Sigma_v : \begin{cases} \dot{v} &= A_v v + B_v e\,, \\ y_v &= C_v v + D_v e\,, \end{cases} \tag{37}$$

  with $\tau = y_w + y_v = Rw + C_v v + D_v e$.
- If we are able to find A_v, B_v, C_v, D_v that guarantee RIS, then RAT will follow and Σ_K will be a *robust controller*.

Construction of a robust controller

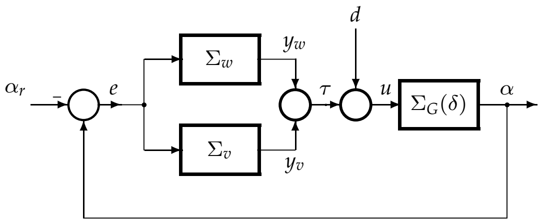

Figure 3: Error feedback control system $\Sigma_e(\delta)$

- For the controller (36) RIS takes the form

$$\sigma\left(\begin{bmatrix} A(\delta)+B(\delta)D_vC & B(\delta)R & B(\delta)C_v \\ QC & P & 0 \\ B_vC & 0 & A_v \end{bmatrix}\right) \subset \mathbb{C}_- \,, \quad \Delta(\delta) \in \Delta_c \,, \tag{38}$$

which is equivalent to say that the unforced closed loop system

$$\Sigma_{uf}(\delta): \begin{bmatrix} \dot{x} \\ \dot{w} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A(\delta)+B(\delta)D_vC & B(\delta)R & B(\delta)C_v \\ QC & P & 0 \\ B_vC & 0 & A_v \end{bmatrix} \begin{bmatrix} x \\ w \\ v \end{bmatrix}, \tag{39}$$

is asymptotically stable for all $\Delta(\delta) \in \Delta_c$.

Uncertain modified plant

- Introduce the *uncertain modified plant* $\Sigma_m(\delta)$ as in Figure 4.

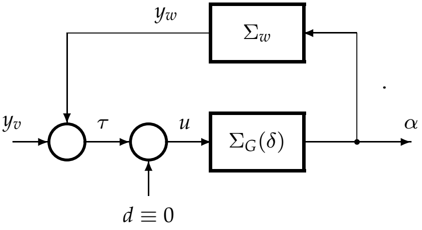

Figure 4: Uncertain modified plant $\Sigma_m(\delta)$

- $\Sigma_m(\delta)$ has order $n_m = 7$ and is described by

$$\Sigma_m(\delta): \left[\begin{array}{c} \dot{\xi} \\ \hline \alpha \end{array}\right] = \left[\begin{array}{c|c} A_m(\delta) & B_m(\delta) \\ \hline C_m & 0 \end{array}\right] \left[\begin{array}{c} \xi \\ \hline y_v \end{array}\right], \tag{40}$$

where $\xi = \begin{bmatrix} x \\ w \end{bmatrix}$ and $\left[\begin{array}{c|c} A_m(\delta) & B_m(\delta) \\ \hline C_m & 0 \end{array}\right] = \left[\begin{array}{cc|c} A(\delta) & B(\delta)R & B(\delta) \\ QC & P & 0 \\ \hline C & 0 & 0 \end{array}\right]$.

Construction of a robust controller

- $\Sigma_{uf}(\delta)$ is an interconnection of $\Sigma_m(\delta)$ and the *output feedback subcontroller* Σ_v of order $n_K - 2$, as it is shown in Figure 5.

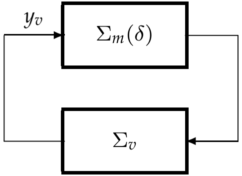

Figure 5: $\Sigma_{uf}(\delta)$ as interconnection of $\Sigma_m(\delta)$ and Σ_v

- We derive the subcontroller Σ_v such that $\Sigma_{uf}(\delta)$ is internally stable for all $\Delta(\delta) \in \Delta_c$ and for this we need the controllability and observability of $\Sigma_m(\delta)$.

- The required controllability and observability of $\Sigma_m(\delta)$ for all $\Delta(\delta) \in \Delta_c$ follow from

$$\det(W_m(\delta)) = \frac{k^3(\delta_k)(\omega_r^2+1)^2}{I^7(\delta_I)p^5(\delta_p)}\left(b^2(\delta_b)\omega_r^2 + (k(\delta_k) - p(\delta_p)\omega_r^2)^2\right) \neq 0\,, \tag{41}$$

and

$$\det(V_m(\delta)) = -\frac{k^3(\delta_k)}{I^5(\delta_I)p^3(\delta_p)}\left(b^2(\delta_b)\omega_r^2 + (k(\delta_k) - p(\delta_p)\omega_r^2)^2\right) \neq 0\,. \tag{42}$$

Construction of a robust controller

- We distinguish the case of $\Sigma_m(\delta)$ *without uncertainties* by setting $\delta = (\delta_k, \delta_I, \delta_p, \delta_b) = 0$.
- In this case the modified plant is denoted by $\Sigma_m(0)$ and called the *nominal modified plant*.
- It is described as

$$\Sigma_m(0) : \left[\begin{array}{c} \dot{\xi} \\ \hline \alpha \end{array}\right] = \left[\begin{array}{c|c} A_m(0) & B_m(0) \\ \hline C_m & 0 \end{array}\right] \left[\begin{array}{c} \xi \\ \hline y_v \end{array}\right]. \tag{43}$$

- For $\Sigma_m(0)$ we construct a classic stabilizing controller (A_v, B_v, C_v, D_v) based on the *full order Luenberger state observer*

$$\dot{\tilde{\xi}} = (A_m(0) - LC_m)\tilde{\xi} + B_m(0)y_v + L\alpha\,, \tag{44}$$

with $\tilde{\xi} = \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix}$

- The *output injection* gain $L \in \mathbb{R}^{7\times 1}$ is chosen such that $\sigma(A_m(0) - LC_m) \subset \mathbb{C}_-$.
- Then we implement the *feedback control law* $y_v = -F\tilde{\xi}$, with the *state feedback* gain matrix $F \in \mathbb{R}^{1\times 7}$ satisfying $\sigma(A_m(0) - B_m(0)F) \subset \mathbb{C}_-$.

Construction of a robust controller

- The resulting unforced closed loop system $\Sigma_{uf}(0)$ with the nominal modified plant $\Sigma_m(0)$ is *internally stable*.

- Finally, we obtain the *subcontroller* Σ_v in the form

$$\Sigma_v : \left[\begin{array}{c} \dot{\tilde{\xi}} \\ \hline y_v \end{array}\right] = \left[\begin{array}{c|c} A_m(0) - LC_m - B_m(0)F & L \\ \hline -F & 0 \end{array}\right] \left[\begin{array}{c} \tilde{\xi} \\ \hline \alpha \end{array}\right], \tag{45}$$

i.e. $$v = \tilde{\xi}\,, \quad A_v = A_m(0) - LC_m - B_m(0)F\,, \quad B_v = L\,, \quad C_v = -F\,, \quad D_v = 0\,, \tag{46}$$

and the *controller* Σ_K which guarantees the *internal stability* and the *asymptotic tracking* of the feedback error control system $\Sigma_e(0)$ with the nominal plant $\Sigma_G(0)$.

- Recall that if this controller satisfies RIS, then it also satisfies RAT.

- In the next section we will show how to examine if this Σ_K guarantees the *internal stability* of the feedback error control system with the *uncertain plant* $\Sigma_G(\delta)$ for all $\Delta(\delta) \in \Delta_c$.

Modelling the uncertain plant

- We apply the robust control theory to analyze the robustness of the controller Σ_K by deriving a test based on the structured singular value.
- For this purpose we will first develop a suitable model of the uncertain plant $\Sigma_G(\delta)$.
- The diagram shown in Figure 6 corresponds to the state space model (11) of the *uncertain plant* $\Sigma_G(\delta)$.

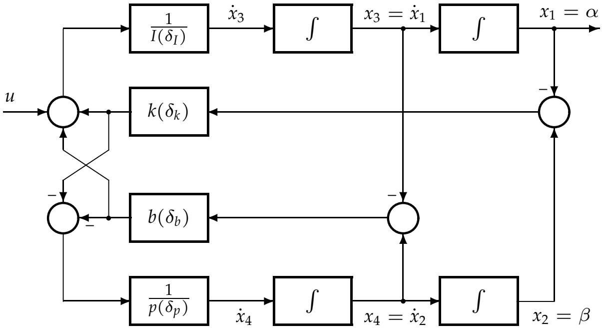

Figure 6: Block diagram of the plant $\Sigma_G(\delta)$

Modelling the uncertain plant

- Using the additive formulas for uncertain parameters we can transform the diagram from Figure 6 to the form shown in Figure 7.

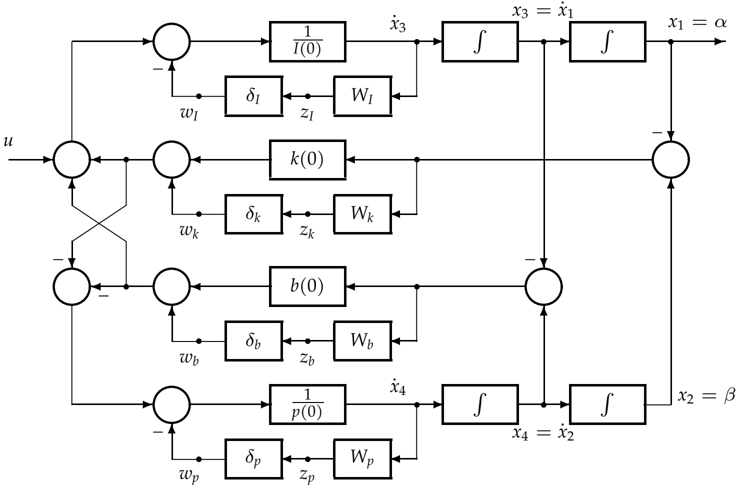

Figure 7: Block diagram of the plant $\Sigma_G(\delta)$ with normalized parametric uncertainties

Modelling the uncertain plant

- In the latter diagram we introduce four *fictitious signals* z_k, z_b, z_I, z_p, entering the uncertainties $\delta_k, \delta_b, \delta_I, \delta_p$ and four *fictitious signals* w_k, w_b, w_I, w_p, leaving uncertainties.
- If we *cut out* all uncertainties, then we obtain a state space model of a system $\Sigma^{\Delta}_{G(0)}$ with inputs w_k, w_b, w_I, w_p, u and outputs $z_k, z_b, z_I, z_p, \alpha$.
- The system $\Sigma^{\Delta}_{G(0)}$ is called the *uncertain plant without uncertainties* and is given by

$$\Sigma^{\Delta}_{G(0)} : \begin{cases} \dot{x}_1 &= x_3 \,, \\ \dot{x}_2 &= x_4 \,, \\ \dot{x}_3 &= \dfrac{k(0)}{I(0)}(x_2 - x_1) + \dfrac{b(0)}{I(0)}(x_4 - x_3) + \dfrac{1}{I(0)}w_k + \dfrac{1}{I(0)}w_b - \dfrac{1}{I(0)}w_I + \dfrac{1}{I(0)}u \,, \\ \dot{x}_4 &= -\dfrac{k(0)}{p(0)}(x_2 - x_1) - \dfrac{b(0)}{p(0)}(x_4 - x_3) - \dfrac{1}{p(0)}w_k - \dfrac{1}{p(0)}w_b - \dfrac{1}{p(0)}w_p \,, \\ z_k &= W_k(x_2 - x_1) \,, \\ z_b &= W_b(x_4 - x_3) \,, \\ z_I &= W_I\Big(\dfrac{k(0)}{I(0)}(x_2 - x_1) + \dfrac{b(0)}{I_(0)}(x_4 - x_3) + \dfrac{1}{I(0)}w_k + \dfrac{1}{I(0)}w_b - \dfrac{1}{I(0)}w_I + \dfrac{1}{I(0)}u\Big) \,, \\ z_p &= W_p\Big(- \dfrac{k(0)}{p(0)}(x_2 - x_1) - \dfrac{b(0)}{p(0)}(x_4 - x_3) - \dfrac{1}{p(0)}w_k - \dfrac{1}{p(0)}w_b - \dfrac{1}{p(0)}w_p\Big) \,, \\ \alpha &= x_1 \,. \end{cases} \tag{47}$$

Modelling the uncertain plant

- $\Sigma^{\Delta}_{G(0)}$ can be written in the matrix form

$$\Sigma^{\Delta}_{G(0)} : \left[\begin{array}{c} \dot{x} \\ \hline z_{\Delta} \\ \hline \alpha \end{array}\right] = \left[\begin{array}{c|c|c} A(0) & B_1 & B(0) \\ \hline C_W & D_W & E_W \\ \hline C & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline w_{\Delta} \\ \hline u \end{array}\right], \tag{48}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, z_{\Delta} = \begin{bmatrix} z_k \\ z_b \\ z_I \\ z_p \end{bmatrix}, w_{\Delta} = \begin{bmatrix} w_k \\ w_b \\ w_I \\ w_p \end{bmatrix}, \left[\begin{array}{c|c|c} A(0) & B_1 & B(0) \\ \hline C_W & D_W & E_W \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} A(0) & B_1 & B(0) \\ \hline WC_1 & WB_1 & WB(0) \\ \hline C & 0 & 0 \end{array}\right], \tag{49}$$

with explicit formulas

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{I(0)} & \frac{1}{I(0)} & -\frac{1}{I(0)} & 0 \\ -\frac{1}{p(0)} & -\frac{1}{p(0)} & 0 & -\frac{1}{p(0)} \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k(0)}{I(0)} & \frac{k(0)}{I(0)} & -\frac{b(0)}{I(0)} & \frac{b(0)}{I(0)} \\ \frac{k(0)}{p(0)} & -\frac{k(0)}{p(0)} & \frac{b(0)}{p(0)} & -\frac{b(0)}{p(0)} \end{bmatrix},$$

$$W = \begin{bmatrix} W_k & 0 & 0 & 0 \\ 0 & W_b & 0 & 0 \\ 0 & 0 & W_I & 0 \\ 0 & 0 & 0 & W_p \end{bmatrix}. \tag{50}$$

Modelling the uncertain plant

- Introducing the *block of uncertainties*

$$\Sigma_{\Delta(\delta)} : \; w_\Delta = \Delta(\delta) z_\Delta \,, \tag{51}$$

we can model $\Sigma_G(\delta)$ as the interconnection shown in Figure 8.

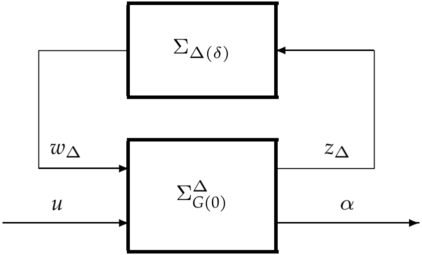

Figure 8: Model of the uncertain plant $\Sigma_G(\delta)$

- For the interconnection, shown in Figure 8, to be well-posed we require

$$\det(I - D_W \Delta(\delta)) \neq 0 \,, \quad \Delta(\delta) \in \Delta_c \,, \tag{52}$$

which holds.

Modelling the uncertain plant

- It is worth to emphasize that in the factorization of the state space matrix of the uncertain plant without uncertainties $\Sigma^{\Delta}_{G(0)}$, i.e.

$$\left[\begin{array}{c|c|c} A(0) & B_1 & B(0) \\ \hline C_W & D_W & E_W \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & W & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{c|c|c} A(0) & B_1 & B(0) \\ \hline C_1 & B_1 & B(0) \\ \hline C & 0 & 0 \end{array}\right], \tag{53}$$

  the first factor matrix

$$\left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & W & 0 \\ \hline 0 & 0 & I \end{array}\right] \tag{54}$$

  depends only on weights, and the second factor matrix

$$\left[\begin{array}{c|c|c} A(0) & B_1 & B(0) \\ \hline C_1 & B_1 & B(0) \\ \hline C & 0 & 0 \end{array}\right] \tag{55}$$

  depends only on nominal parameters. We can also assume that $\det W \neq 0$, which is equivalent to the fact that all four parameters k, I, p and b are allowed to be uncertain.

- If not all of them are uncertain the model $\Sigma^{\Delta}_{G(0)}$ has to be appropriately modified.

Control system with the uncertain plant

- Since the uncertain plant $\Sigma_G(\delta)$ is modelled as in Figure 8, then the feedback error control system can be reshaped as in Figure 9.

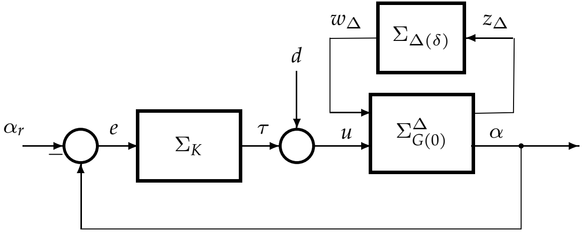

Figure 9: Model of the control system with an uncertain plant $\Sigma_G(\delta)$

- Recall that Σ_K has been designed to stabilize the nominal plant $\Sigma_G(0)$, which means that

$$\sigma\left(\begin{bmatrix} A(0)+B(0)D_KC & B(0)C_K \\ B_KC & A_K \end{bmatrix}\right) \in \mathbb{C}_- \,. \tag{56}$$

Control system with the uncertain plant

- In order analyze RIS we follow the classic way developed within the robust control theory.
- Notice that if $\alpha_r = 0$, then $\Sigma_{uf}(\delta)$ from Figure 9 is an interconnection of some system Σ_M and $\Sigma_{\Delta(\delta)}$ as it is shown in Figure 10.

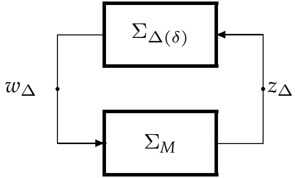

Figure 10: $\Sigma_{uf}(\delta)$ as an interconnection of Σ_M and $\Sigma_{\Delta(\delta)}$

- Simple computations show that Σ_M is described by

$$\Sigma_M : \left[\begin{array}{c} \dot{x} \\ \dot{x}_K \\ \hline z_\Delta \end{array}\right] = \left[\begin{array}{cc|c} A(0)+B(0)D_KC & B(0)C_K & B_1 \\ B_KC & A_K & 0 \\ \hline C_W+E_WD_KC & E_WC_K & D_W \end{array}\right] \left[\begin{array}{c} x \\ x_K \\ \hline w_\Delta \end{array}\right], \quad \left[\begin{array}{c} x(0) \\ x_K(0) \end{array}\right] = \left[\begin{array}{c} x_0 \\ x_{K0} \end{array}\right]. \tag{57}$$

- Although the internal stability of $\Sigma_{uf}(\delta)$ is essentially a state space concept it can be examined by using transfer functions of the systems involved instead of their state space models. However, for such an analysis the state space models have to be stabilizable and detectable and the loop in Figure 10 has to be modified to the form shown in Figure 11.

Control system with the uncertain plant

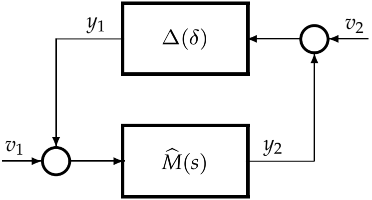

Figure 11: Interconnection of $\widehat{M}(s)$ and $\Delta(\delta)$ for examination of the internal stability

- In Figure 11 $\widehat{M}(s)$ denotes the transfer function of Σ_M and is given by

$$\widehat{M}(s) = W \left(\begin{array}{cc|c} A(0) & B(0)C_K & B_1 \\ B_K C & A_K & 0 \\ \hline C_1 & B(0)C_K & B_1 \end{array} \right) = W\widehat{M}_0(s)\,. \qquad (58)$$

- The matrix W allows to scale the "size" of the transfer function $\widehat{M}(s)$.
- $\widehat{M}(s)$ and $\widehat{M}_0(s)$ are stable (in the BIBO sense). $\Delta(\delta)$ is a static matrix. $\det W \neq 0$ implies that Σ_M is stabilizable and detectable if and only if the state space realization of $\widehat{M}_0(s)$ is so.

Robust internal stability

- For the interconnection in Figure 11 we get

$$\begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{bmatrix} = \begin{bmatrix} \Delta(\delta)(I - \widehat{M}(s)\Delta(\delta))^{-1}\widehat{M}(s) & \Delta(\delta)(I - \widehat{M}(s)\Delta(\delta))^{-1} \\ (I - \widehat{M}(s)\Delta(\delta))^{-1}\widehat{M}(s) & (I - \widehat{M}(s)\Delta(\delta))^{-1}\widehat{M}(s)\Delta(\delta) \end{bmatrix} \begin{bmatrix} \hat{v}_1(s) \\ \hat{v}_2(s) \end{bmatrix} \tag{59}$$

  and RIS holds if and only if all the four transfer functions in (59) are proper and stable for all $\Delta(\delta) \in \Delta_c$.
- Since $\Delta(\delta)$ and $\widehat{M}(s)$ are proper and stable we immediately get:

Lemma 4.1

$\Sigma_e(\delta)$ satisfies RIS if and only if

$$(I - \widehat{M}(s)\Delta(\delta))^{-1} \in \mathcal{RH}_\infty\,, \quad \Delta(\delta) \in \Delta_c\,. \tag{60}$$

- (60) is very hard to check and the following *necessary and sufficient* result makes life easier.

Lemma 4.2

The condition (60) *is satisfied and, consequently, $\Sigma_e(\delta)$ satisfies RIS if and only if*

$$\det(I - \widehat{M}(j\omega)\Delta(\delta)) \neq 0\,, \quad \Delta(\delta) \in \Delta_c\,, \quad \omega \in \mathbb{R}\,. \tag{61}$$

Robust internal stability

- The main problem with (61) it that it has to be checked for all matrices $\Delta(\delta) \in \Delta_c$ and all $\omega \in \mathbb{R}$.
- The concept of a *structured singular value* turns out to be helpful since it allows to replace (61) by a much more practical but still equivalent condition.

Definition 4.3

Let $\omega \in \mathbb{R}$. **The structured singular value** $\mu_{\Delta_c}(\widehat{M}(j\omega))$ of a matrix $\widehat{M}(j\omega)$ for the uncertainty structure set Δ_c is defined by the expression

$$\mu_{\Delta_c}(\widehat{M}(j\omega)) := \frac{1}{\gamma^*} = \frac{1}{\sup\{\gamma : \ \det(I - \widehat{M}(j\omega)\Delta(\delta)) \neq 0 \,, \quad \Delta(\delta) \in \gamma\Delta_c\}} \,. \tag{62}$$

- Since the structure set Δ_c is star-shaped, then for $0 < \gamma_1 \leq \gamma_2$ we have

$$\gamma_1 \Delta_c \subset \gamma_2 \Delta_c \,. \tag{63}$$

- For $\gamma \leq \gamma^*$ we have

$$\det(I - \widehat{M}(j\omega)\Delta(\delta)) \neq 0 \,, \quad \Delta(\delta) \in \gamma\Delta_c \,. \tag{64}$$

Robust internal stability - main result

Theorem 4.4

The error feedback control system $\Sigma_e(\delta)$ satisfies RIS if and only if the structured singular value of the matrix $\widehat{M}(j\omega)$ for the structure set Δ_c satisfies

$$\mu_{\Delta_c}(\widehat{M}(j\omega)) \leq 1\,, \quad \omega \in \mathbb{R}\,. \tag{65}$$

- For the structured singular value we have

$$\gamma\, \mu_{\Delta_c}(\widehat{M}(j\omega)) = \mu_{\Delta_c}(\gamma \widehat{M}(j\omega)) = \mu_{\gamma\Delta_c}(\widehat{M}(j\omega))\,, \tag{66}$$

  which means that scaling μ by the factor γ is equivalent to scaling $\widehat{M}(j\omega)$ or Δ_c.
- In practice we compute only some *maximum bound* γ_u of μ, i.e.

$$\sup_{\omega\in\mathbb{R}} \mu_{\Delta_c}(W\widehat{M}_0(j\omega)) \leq \gamma_u\,, \tag{67}$$

  and then conclude RIS of $\Sigma_e(\delta)$ for the scaled (new) matrix of weights $\frac{1}{\gamma_u}W$.
- The Robust Control Toolbox of the MATLAB package has a function `mussv` which returns a series of the lower and the upper estimates of the structured singular value

$$\gamma_l(\omega_i) \leq \mu_{\Delta_c}(\widehat{M}(j\omega_i)) \leq \gamma_u(\omega_i)\,, \quad (\omega_i)_{i=0}^{i=N} \subset [0, \infty)\,, \tag{68}$$

  where the values of ω_i are adaptively selected by MATLAB.

Numerical example - nominal plant

- Parameters of the nominal plant $\Sigma_G(0)$ and the reference signal $\alpha_r(t)$

$$k(0) = 750 \left[\frac{\mathrm{N \cdot m}}{\mathrm{rad}}\right], \quad b(0) = 0.01 \; [\mathrm{N \cdot m \cdot s}], \quad I(0) = 1.7 \; [\mathrm{kg \cdot m^2}], \quad p(0) = 0.1 \; [\mathrm{kg \cdot m^2}], \tag{69}$$

for the reference $\alpha_r = a\sin(\omega_r t)$ and the disturbance d_0

$$a = 1 \; [\mathrm{rad}], \quad \omega_r = 1 \left[\frac{\mathrm{deg}}{\mathrm{s}}\right] = \frac{\pi}{180} \left[\frac{\mathrm{rad}}{\mathrm{s}}\right], \quad d_0 = 0.01 \; [\mathrm{N \cdot m}], \tag{70}$$

- The nominal plant $\Sigma_G(0)$

$$\left[\begin{array}{c|c} A(0) & B(0) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -441.17 & 441.17 & -0.0059 & 0.0059 & 0.5882 \\ 7500 & -7500 & 0.1 & -0.1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{71}$$

- The real parameters k, b, I and p belong to the intervals

$$k \in (k_{\min}, k_{\max}), \quad b \in (b_{\min}, b_{\max}), \quad I \in (I_{\min}, I_{\max}), \quad p \in (p_{\min}, p_{\max}), \tag{72}$$

where

$$\begin{array}{ll} k_{\min} = 600, & k_{\max} = 900, \\ b_{\min} = 0.007, & b_{\max} = 0.013, \\ I_{\min} = 1.53, & I_{\max} = 1.87, \\ p_{\min} = 0.095, & p_{\max} = 0.105, \end{array} \tag{73}$$

Numerical example - nominal plant

- The weight matrix W

$$W = \begin{bmatrix} W_k & 0 & 0 & 0 \\ 0 & W_b & 0 & 0 \\ 0 & 0 & W_I & 0 \\ 0 & 0 & 0 & W_p \end{bmatrix} = \begin{bmatrix} 150 & 0 & 0 & 0 \\ 0 & 0.003 & 0 & 0 \\ 0 & 0 & 0.17 & 0 \\ 0 & 0 & 0 & 0.005 \end{bmatrix}, \tag{74}$$

- The feedback gain F

$$F = \begin{bmatrix} 37.0562 & -18.4681 & 11.6181 & -2.2908 & 4.3166 & 8.1139 & 8.4203 \end{bmatrix}.$$

- The output injection L

$$L = \begin{bmatrix} 2.8190 \\ 2.7162 \\ 3.8733 \\ 3.7738 \\ 7.3731 \\ 4.6268 \\ 2.0938 \end{bmatrix}.$$

Numerical example - nominal plant

- Final controller Σ_K

$$\left[\begin{array}{cccccccccc|c}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -0.0003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2.8 & 0 & 1 & 0 & 0 & 0 & 0 & 2.8190 \\
0 & 0 & 0 & -2.7 & 0 & 0 & 1 & 0 & 0 & 2.7162 & \\
0 & 0 & 0 & -466.8 & 452 & -6.8 & 1.4 & -2 & -4.8 & -5 & 3.8733 \\
0 & 0 & 0 & 7495.2 & -7500 & 0.1 & -0.1 & 0 & 0 & 0 & 3.7738 \\
0 & 0 & 0 & -6.4 & 0 & 0 & 0 & 0 & 1 & 0 & 7.3731 \\
0 & 0 & 0 & -3.6 & 0 & 0 & 0 & 0 & 0 & 1 & 4.6268 \\
0 & 0 & 0 & -1.1 & 0 & 0 & -0 & 0 & -0.0003 & 0 & 2.0938 \\
\hline
1 & 0 & 0 & -37.0562 & 18.4681 & -11.6181 & 2.2908 & -4.3166 & -8.1139 & -8.4203 & 0
\end{array}\right]$$

Numerical example - nominal plant

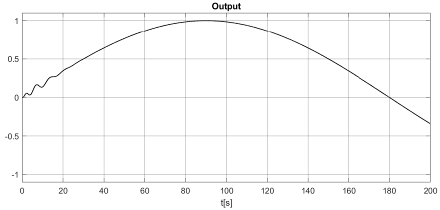

Figure 12: The output $\alpha(t)$ for $\Sigma_G(0)$

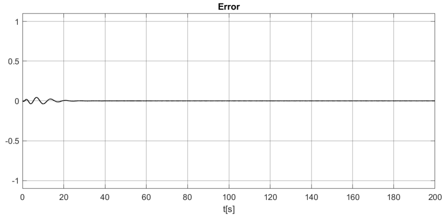

Figure 13: The error $e(t)$ for $\Sigma_G(0)$

Numerical example - nominal plant

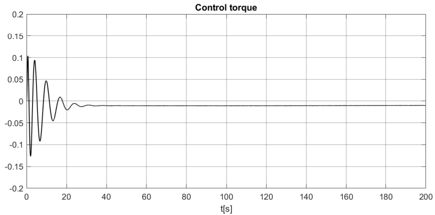

Figure 14: The control torque $\tau(t)$ for $\Sigma_G(0)$

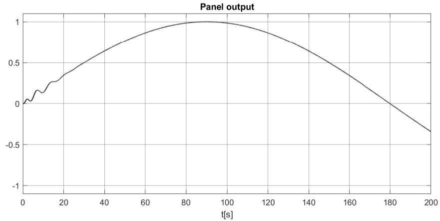

Figure 15: The panel output $\beta(t)$ for $\Sigma_G(0)$

Numerical example - uncertain plant

- We used the *mussv* MATLAB procedure to compute the maximum upper bound γ_u

$$\sup_{\omega\geq 0} \mu_{\Delta_c}(\widehat{M}(j\omega)) \leq \gamma_u \tag{75}$$

and obtained

$$\gamma_u = 1.0190 \quad \text{for} \quad \omega = 1.324\,, \tag{76}$$

with

$$1.0164 = \gamma_l \leq \mu_{\Delta_c}(\widehat{M}(j1.324)) \leq \gamma_u = 1.0190\,. \tag{77}$$

- The weight matrix W rescaled by the factor $\gamma = 1.02 \geq \gamma_u$

$$W_\gamma = \gamma^{-1} W = \begin{bmatrix} 147.0588 & 0 & 0 & 0 \\ 0 & 0.0029 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 \\ 0 & 0 & 0 & 0.0049 \end{bmatrix} \tag{78}$$

- New (rescaled) intervals

$$\begin{array}{ll} k_{\min} = 602.9412\,, & k_{\max} = 897.0588\,, \\ b_{\min} = 0.0071\,, & b_{\max} = 0.0129\,, \\ I_{\min} = 1.5333\,, & I_{\max} = 1.8667\,, \\ p_{\min} = 0.0951\,, & p_{\max} = 0.1049\,. \end{array} \tag{79}$$

- The controller will robustly stabilize all plants $\Sigma_G(\delta)$ with real parameters k, b, I and p from these new intervals and, moreover, the robust asymptotic tracking condition will hold.

Numerical example - uncertain plant

- For the plant $\Sigma_G(\delta)$ with the parameters

$$k = 617.6471, \quad b = 0.0074, \quad I = 1.85, \quad p = 0.1044, \tag{80}$$

the state space matrix takes the form

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C & 0 \end{array}\right] := \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -333.9 & 333.9 & -0.004 & 0.004 & 0.5405 \\ 5915.5 & -5915.5 & 0.0704 & -0.0704 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \tag{81}$$

Numerical example - uncertain plant

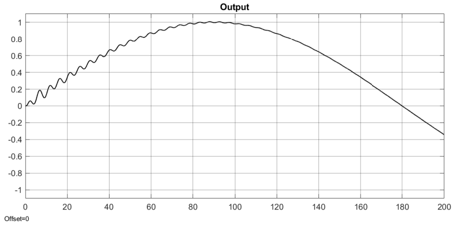

Figure 16: The output $\alpha(t)$ for $\Sigma_G(\delta)$

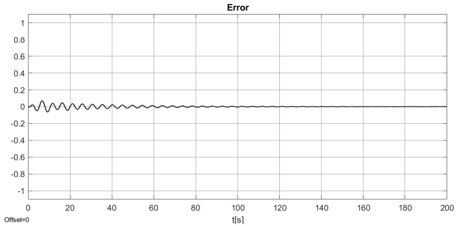

Figure 17: The error $e(t)$ for $\Sigma_G(\delta)$

Numerical example - uncertain plant

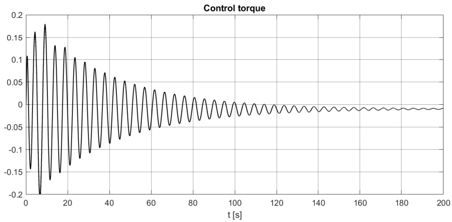

Figure 18: The control torque $\tau(t)$ for $\Sigma_G(\delta)$

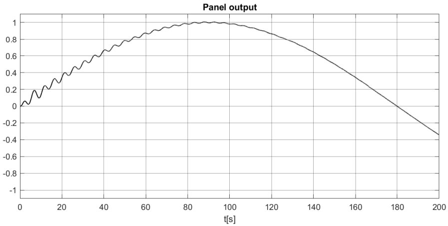

Figure 19: The panel output $\beta(t)$ for $\Sigma_G(\delta)$